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The propagation of detonation waves in channels of varying cross-section

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One-dimensional detonation wave propagation in channels of varying cross-section is reconsidered and studied in detail. Different analytical solutions are given for the case of an accelerated detonation wave in a converging channel and for a decelerated detonation wave in a diverging channel. Separation of the leading shock and the reaction zone in the second case is taken into account. Two- and three-dimensional problems of geometrical detonation wave dynamics can be solved by adapting the well-known approach of Whitham, but Whitham's method is based on a suitable one-dimensional analytical model.

1. Introduction

Multi-dimensional problems of detonation-wave propagation can successfully be overcome by applying the well-known theory of geometrical shock dynamics of Whitham (1957, 1959, 1974). This approach has been demonstrated by the work of Edwards, Thomas & Nettleton (1979), Thomas (1979), and more recently by Bartlmä & Schröder (1986). For certain other cases, a modification of Whitham's theory is necessary. In all cases, however, a suitable analytical model of the one-dimensional propagation of shock and detonation waves in channels of varying cross-section can provide the basis of the theory.

The work of Chester (1953) and Chisnell (1957) can be applied to the propagation of ordinary shock waves in channels of varying cross-section with good results. Several papers have been published with regard to detonation wave propagation. Teipel (1975, 1976) and Schnitzspan (1976) assumed the existence of a Chapman–Jouguet detonation (CJ-detonation) at all times, despite the area variation in the channel. This assumption is only true for a very special relationship between the heat production by the chemical reaction and the area change, but no burning reaction is capable of satisfying such a condition. Only at this price can a self-similar solution be achieved.

Later Teipel (1983) dropped the restriction of a CJ-detonation and applied the so-called characteristic rule (see Whitham 1974). He has given a complete numerical solution and also derived a functional relation between the area change in the channel and the pressure rise at the detonation wave. This is analogous to the relation derived by Chester (1953) and Chisnell (1957) for ordinary shock waves. Unfortunately, his analytic solution is only valid for relatively small amounts of heat production per unit mass of the burning mixture, a restriction which is unrealistic for detonation waves. Furthermore, only accelerated detonation waves were treated since only converging cylindrical or spherical flow was considered. None of the aforementioned papers has taken into account the important case of a decelerated

CJ-detonation in a diverging channel. This phenomenon differs radically from the accelerated detonation wave, as will be discussed later.

For this reason and also because of its importance for the theory of geometrical detonation wave dynamics, the one-dimensional problem of detonation wave propagation in channels of varying cross-section is reconsidered here. An improved analytic solution for arbitrary heat addition in the case of an accelerated detonation wave in a converging tube is given. The solution presented here can provide the basis for the calculation of two- or three-dimensional detonation wave propagation, a topic which will be the subject of another paper. Finally, the shortcomings and limitations of the Whitham theory applied to detonation waves are discussed.

2. The detonation wave models

Ignoring the complex three-dimensional unsteady inner structure, a detonation wave may be thought of as a reaction zone in which a fast exothermic chemical reaction occurs, preceded by a shock wave. The burning reaction is triggered after a certain induction time by the strong pressure and temperature rise in the shock wave, whereas the latter is supported by the heat liberated in the reaction zone.

Different detonation wave models are available, depending upon the desired accuracy and the specific problem encountered. Some of the more pertinent models are briefly described below.

2.1. *The single-front model*

Usually, the chemical reaction zone will be very thin. Furthermore, the thickness of the induction zone will decrease exponentially with increasing temperature and may be ignored in a great many cases. Under these assumptions, the leading shock wave and the reaction zone are considered as a single discontinuity, giving rise to the single-front model (figure 1*a*). This is the simplest theoretical description of a detonation wave.

This model is described by the well-known reaction wave equations, which are discussed in detail and given for unsteady waves by, among others, Oppenheim & Stern (1959), Cherny (1973) and by the present author (Bartlmä 1971, 1975). A somewhat simplified form will be employed here because, in general, the detonation wave Mach numbers are of the order of 5 or more for hydrocarbon–air mixtures, and up to the order of about 10 for hydrogen–oxygen mixtures. The dimensionless value of the heat liberated by the chemical reaction, Q, per unit mass of the mixture is also roughly of the same order of magnitude. Hence for $M^2 \gg 1$ and $Q \sim M$, the following slightly simplified reaction wave equations have been derived:

$$\left.\begin{aligned} \frac{u}{c_0} &\approx \frac{1}{(\gamma_2+1)} M(1+\Omega^{\frac{1}{2}}), \\ \frac{p}{p_0} &\approx \frac{\gamma_1}{(\gamma_2+1)} M^2(1+\Omega^{\frac{1}{2}}), \\ \frac{\rho_0}{\rho} &\approx 1-\frac{1}{(\gamma_2+1)}(1+\Omega^{\frac{1}{2}}), \\ \frac{c_{p2}T}{c_{p1}T_0} &\approx \frac{(\gamma_1-1)}{(\gamma_2-1)}\left(\frac{c}{c_0}\right)^2 = \frac{(\gamma_1-1)}{(\gamma_2-1)}\frac{\gamma_2}{\gamma_1}\frac{p}{p_0}\frac{\rho_0}{\rho}, \end{aligned}\right\} \qquad (2.1)$$

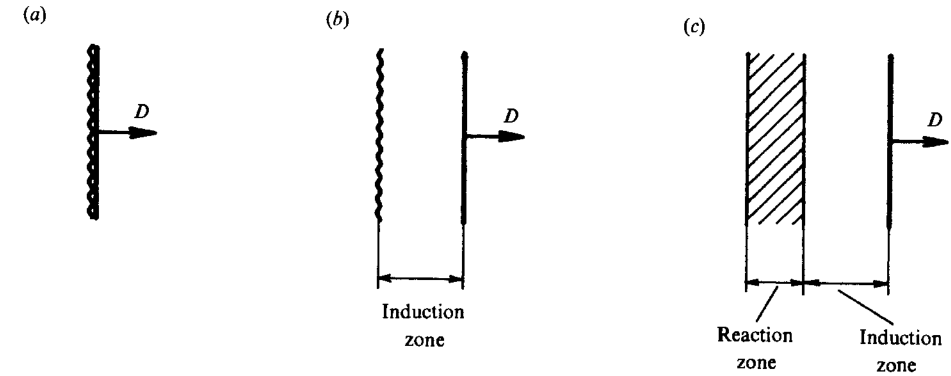

FIGURE 1. Detonation wave models: (*a*) single-front model, (*b*) two-front model, (*c*) refined two-front model.

where
$$M = D/c_0,$$
$$\Omega = 1-2\left[\frac{\gamma_2}{\gamma_1}+(\gamma_2+1)\,Q\right]\frac{1}{M^2},$$
$$Q = \left[\frac{(\gamma_2-1)}{(\gamma_1-1)}(1+Q_0)-\frac{\gamma_2}{\gamma_1}\right], \quad Q_0 = \frac{q}{c_{p1}T_0}.$$

For $M_{CJ} \leqslant M < \infty$ we have $0 \leqslant \Omega < 1$. In the above equations p, ρ, T and c are the pressure, density, temperature and sound velocity; c_{p1} is the specific heat at constant pressure, of the unburned gas. Ideal gas was assumed at both sides of the front but the ratio of the specific heats was allowed to change from γ_1 to γ_2 across the detonation wave. Q may be considered as an effective heat addition, which reduces to Q_0 if γ is kept constant. Finally u and D are the flow velocity of the reaction products and the detonation velocity in a fixed coordinate system. The unburned gas is at rest.

The detonation wave Mach number M has a lower limit of M_{CJ} which corresponds to the case of the steady CJ-detonation. The term Ω in (2.1) becomes zero, a characteristic feature of this special case which reduces the reaction wave equations to an exceptionally simple form. As $M \to \infty$, then $\Omega \to 1$, and neglecting the change in γ, the equations (2.1) become the equations for an unsteady strong shock wave:

$$\left.\begin{aligned} \frac{u}{c_0} &\approx \frac{2}{\gamma_2+1}M, \quad \frac{p}{p_0} \approx \frac{2\gamma_1}{\gamma_2+1}M^2, \quad \frac{\rho_0}{\rho} \approx \frac{\gamma_2-1}{\gamma_2+1}, \\ \frac{c}{c_0} &\approx \frac{[2\gamma_2(\gamma_2-1)]^{\frac{1}{2}}}{(\gamma_2+1)}M, \quad \frac{c_{p2}T}{c_{p1}T_0} \approx \frac{2\gamma_2(\gamma_1-1)}{(\gamma_2+1)^2}M^2. \end{aligned}\right\} \qquad (2.2)$$

From (2.2), it follows that in the case of a strongly overdriven detonation wave, the influence of the chemical reaction is of second order and therefore may be neglected. Experiments clearly show this tendency.

2.2. *The two-front model*

The quite simple single-front model has proved to be extremely useful. Nevertheless, there exist cases when the induction time plays an important part, especially if decelerated detonation waves are involved. Hence, the detonation wave model must be modified by considering the shock and the reaction zone as two different discontinuities separated by an induction zone (figure 1*b*). This description leads to a two-front model.

The induction zone thickness is governed by the ignition delay time τ which strongly depends on the temperature. Since there is no satisfactory theory available, τ must be given by an empirical relation for the gas mixture under consideration.

A further refinement of the two-front model, such as considering a finite reaction rate which specifies a finite reaction zone thickness (figure 1*c*), does not seem to be feasible for this particular problem, since the detonation wave structure is also not considered.

Here it is important to remember that a detonation wave actually has quite a complex, three-dimensional, unsteady inner structure consisting of local detonation waves, as described above, and ordinary shock waves (see e.g. Strehlow 1984; Fickett & Davis 1979). However, despite the simple models used, calculated results of the detonation velocities may be considered as good approximations to actual values.

3. The propagation of detonation waves in a converging channel

As previously mentioned, there are distinct behavioural differences between accelerated and decelerated detonation waves. Consequently, different mathematical models are required depending upon the movement of the detonation wave in a converging or diverging duct, particularly if the incident wave is a steady CJ-detonation.

Consider a slightly overcompressed detonation wave originally moving with a Mach number of M_0 down a tube of constant cross-section A_0. At $x = 0$ the channel starts to converge, and as a result the detonation wave is accelerated. One-dimensional flow of an ideal gas is assumed. The unburned gas mixture in front of the detonation wave is at rest. The area change is assumed to be small: $(A(x)-A_0)/A_0 \ll 1$. Then, the dependence of the detonation wave Mach number has to be determined for a given cross-section distribution $A(x)$ for $x > 0$.

Under the assumptions made above, the motion of the burned gas is governed by the following equations for the conservation of mass, momentum, and energy:

$$\left.\begin{aligned}
&\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+\rho\frac{\partial u}{\partial x}+\frac{\rho u}{A(x)}\frac{\mathrm{d}A(x)}{\mathrm{d}x}=0,\\
&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0,\\
&\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}-c^2\left(\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}\right)=0.
\end{aligned}\right\} \qquad (3.1)$$

In addition, there are some initial and boundary conditions at the detonation wave. A schematic picture of the (x, t)-diagram for the detonation wave propagation is shown in the upper part of figure 2.

The undisturbed state of the unburned gas ahead of the detonation wave (subscript 0), and the state of the reaction products of the oncoming detonation (subscript 2) are known. If the area change is small, the state of the reaction products behind the disturbed detonation wave (no subscript) will deviate only little from state 2. Therefore, linearization is possible with respect to state 2: $u = u_2+u'$, $p = p_2+p'$, $\rho = \rho_2+\rho'$, ..., where $u' \ll u_2$, $p' \ll p_2$, $\rho' \ll \rho_2$, ...

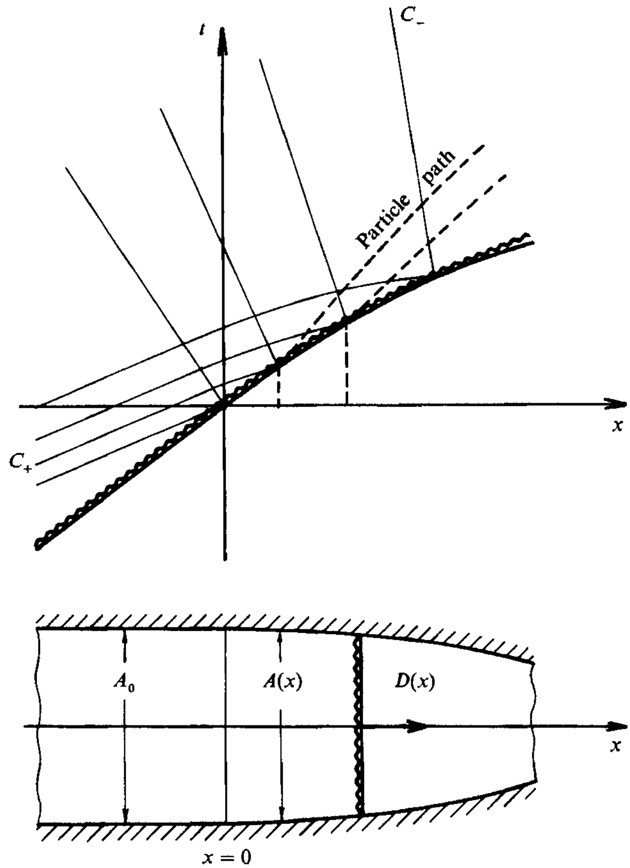

FIGURE 2. Accelerated detonation wave in a converging channel.

From this point on, the steps of the calculation are the same as for the propagation of an ordinary shock wave. Therefore, only a brief outline is given here. Details may be found in Whitham (1974).

The linearized compatibility conditions are derived from (3.1). For the case in question, only the equation for the C_+-family of the characteristics is required. After integration it may be written in following form:

$$(p-p_2)+\rho_2 c_2(u-u_2) = -\rho_2 c_2^2 \frac{u_2}{(u_2+c_2)} \frac{A(x)-A_0}{A_0} + f\{x-(u_2+c_2)t\} \quad \text{along } C_+, \tag{3.2}$$

and for the equation of the C_+ characteristics we get

$$x-(u_2+c_2)t = \text{const.}$$

The arbitrary function, f, follows from the initial conditions:

$$f\{x-(u_2+c_2)t\} = 0. \tag{3.4}$$

This indicates that within the framework of our linearization, no disturbances from downstream occur.

Thus far, the calculations have been identical for both the shock wave and the detonation wave. Now an appropriate detonation wave model must be chosen, which in this case may be the single-wave model. After linearizing the reaction wave equations (2.1), the perturbations of u and p in terms of the change in the detonation wave Mach number, $M' = M - M_0$, are as follows:

$$\left.\begin{aligned} \frac{u'}{c_0} &= \frac{u-u_2}{c_0} = \frac{1}{(\gamma_2+1)}\frac{(1+\Omega_0^{\frac{1}{2}})}{\Omega_0^{\frac{1}{2}}}M', \\ \frac{p}{p_0} &= \frac{p-p_2}{p_0} = \frac{\gamma_1}{(\gamma_2+1)}M_0\frac{(1+\Omega_0^{\frac{1}{2}})}{\Omega_0^{\frac{1}{2}}}M', \end{aligned}\right\} \tag{3.5}$$

where

$$\Omega_0 = 1 - 2\left[\frac{\gamma_2}{\gamma_1} + (\gamma_2+1)\,Q\right]\frac{1}{M^2}.$$

Combining (3.2) and (3.5), the relation between the area change and the Mach-number disturbance becomes

$$\frac{A-A_0}{A_0} = -H(M_0)\,(M-M_0), \tag{3.6}$$

where

$$H(M_0) = \frac{1}{(\gamma_2+1)}\frac{\left(\dfrac{u_2}{c_0}+\dfrac{c_2}{c_0}\right)}{\dfrac{p_2}{p_0}\dfrac{u_2}{c_0}}\left[M_0(1+\Omega_0^{\frac{1}{2}}) + \frac{\rho_2 c_2}{\rho_0 c_0}\right]\frac{(1+\Omega_0^{\frac{1}{2}})}{\Omega_0}. \tag{3.7}$$

So far the calculation has remained consistent with our assumption of small area change. No additional assumptions have been made. Reflected waves are of second order.

Turning now to finite area change, (3.6) may be written in differential form, as has been done in the case of shock wave propagation by Chisnell (1957):

$$\frac{\mathrm{d}A}{A} = -H(M)\,\mathrm{d}M, \tag{3.8}$$

where

$$H(M) = \frac{1}{(\gamma_2+1)}\frac{\left(\dfrac{u}{c_0}+\dfrac{c}{c_0}\right)}{\dfrac{p}{p_0}\dfrac{u}{c_0}}\left[M(1+\Omega^{\frac{1}{2}}) + \frac{\rho\, c}{\rho_0 c_0}\right]\frac{(1+\Omega^{\frac{1}{2}})}{\Omega^{\frac{1}{2}}}. \tag{3.9}$$

The initial conditions are

$$A = A_0: \quad M = M_0, \quad H(M) = H(M_0).$$

The Mach-number function $H(M)$ has following limits:

$$\left.\begin{aligned} &\text{as} \quad M \to M_{\mathrm{CJ}}, \quad \Omega \to 0, \quad H(M_{\mathrm{CJ}}) \to \infty; \\ &\text{as} \quad M \to \infty, \quad \Omega \to 1, \quad H(M) \sim \left[1 + \frac{2}{\gamma_2} + \left(\frac{\gamma_2}{\gamma_2-1}\right)^{\frac{1}{2}}\right]\frac{1}{M}. \end{aligned}\right\} \tag{3.10}$$

The limiting case, $M \to \infty$, is again formally identical with the case of a strong shock wave. In the case of vanishing heat addition ($Q \to 0$) the function $H(M)$ becomes the

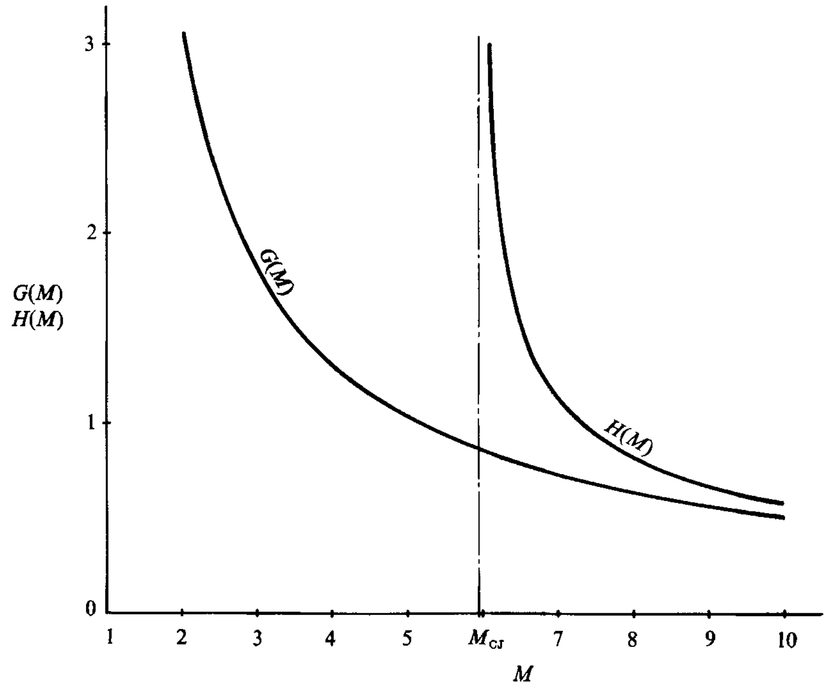

FIGURE 3. $G(M)$ for $Q = 0$ (shock wave) and $H(M)$ for $Q = 7$ (detonation wave) as functions of the wave Mach number.

well-known Mach-number function $G(M)$ given for an ordinary shock wave by Chester (1953) and Chisnell (1957). In the literature $G(M)$ is known as the Chisnell function and may be written in following form:

$$G(M) = \frac{M}{(M^2-1)}\lambda(M), \tag{3.11}$$

where
$$\lambda(M) = \left(1-\frac{2}{\gamma_1+1}\frac{1-\mu^2}{\mu}\right)\left(1+2\mu+\frac{1}{M^2}\right), \tag{3.12}$$

$$\mu = \frac{(\gamma_1-1)M^2+2}{2\gamma_1 M^2-(\gamma_1-1)}, \tag{3.13}$$

$$M = \frac{U}{c_0},$$

and where U is the propagation velocity of the shock wave in a fixed coordinate system.

In figure 3, $G(M)$ for the shock wave ($Q = 0$) and $H(M)$ for a typical detonation wave are drawn as functions of the wave Mach number M. Figure 3 shows clearly that for an accelerated detonation wave it is not permissible to describe the propagation process of a detonation wave by the leading shock wave alone. As we shall see later, this is not necessarily true in the case of a decelerated detonation wave. Furthermore, for highly overcompressed detonation waves, we can see again that the influence of the heat addition vanishes.

Let us recall the assumption $M^2 \gg 1$, which was used for the derivation of $H(M)$. A comparison with the exact expression, derived from the complete reaction wave equations, generally shows that the error will be below 1%. This relation holds for values of the dimensionless heat addition that are common for detonation waves in premixed gases, namely $5 < Q < 10$.

Finally, the desired relation between channel area and detonation wave Mach number can be obtained by integrating (3.8):

$$\frac{A}{A_0} = \exp\left\{-\int_{M_0}^{M} H(M)\,\mathrm{d}M\right\}. \tag{3.14}$$

Performing the integration results in

$$\frac{A}{A_0} = \left(\frac{1-R}{1-R_0}\right)^{\gamma_2+2/2\gamma_2}\left(\frac{1+R_0}{1+R}\right)^{\frac{1}{2}}$$
$$\times\left\{\frac{\dfrac{1}{4}\dfrac{(3-\gamma_2)}{(\gamma_2-1)}+\dfrac{1}{(1-R)}\left[1+\left[\dfrac{\gamma_2}{2(\gamma_2-1)}(1+R)\left(1-\dfrac{1}{\gamma_2}R\right)\right]^{\frac{1}{2}}\right]}{\dfrac{1}{4}\dfrac{(3-\gamma_2)}{(\gamma_2-1)}+\dfrac{1}{(1-R_0)}\left[1+\left[\dfrac{\gamma_2}{2(\gamma_2-1)}(1+R_0)\left(1-\dfrac{1}{\gamma_2}R_0\right)\right]^{\frac{1}{2}}\right]}\right\}^{-[\gamma_2/2(\gamma_2-1)]^{\frac{1}{2}}}$$
$$\times\frac{\exp\left\{\dfrac{1}{\gamma_2^{\frac{1}{2}}}\sin^{-1}\left(\dfrac{\gamma_2-1-2R}{\gamma_2+1}\right)\right\}}{\exp\left\{\dfrac{1}{\gamma_2^{\frac{1}{2}}}\sin^{-1}\left(\dfrac{\gamma_2-1-2R_0}{\gamma_2+1}\right)\right\}}, \tag{3.15}$$

where $R = \Omega^{\frac{1}{2}}, \quad R_0 = \Omega_0^{\frac{1}{2}}.$

Equation (3.15) is only valid, strictly speaking, for $M_0 > M_{CJ}$, since in the vicinity of $M_0 = M_{CJ}$ the linearization in (3.5) is not correct.

The result (3.15) could also have been achieved by applying Whitham's (1974) characteristic rule. But the derivation used here gives better insight and shows more clearly the limits of the method.

As in the case of ordinary shock propagation treated by Chester (1953) and Chisnell (1957), the neglect of secondary shock waves now can only be justified by comparison with experiment. Chisnell's results for shock waves were surprisingly good; a fact still not yet well understood. Since the shock is dominant in a detonation, equally good results are to be expected, especially in the Mach-number region under consideration.

In figure 4, the relation between area ratio A/A_0 and Mach number ratio M/M_0 is shown for different values of M_0. For very high detonation wave Mach numbers ($M \to \infty$), the result becomes independent of the heat liberated by the chemical reaction, and simplifies to

$$\frac{A}{A_0} = \left(\frac{M_0}{M}\right)^n, \tag{3.16}$$

where $$n = 1+\frac{2}{\gamma_2}+\left(\frac{2\gamma_2}{\gamma_2-1}\right)^{\frac{1}{2}}.$$

This relation is the well-known result for strong shock waves. The case of strong imploding shock waves having spherical or cylindrical symmetry has been solved by Guderley (1942) and Butler (1954).

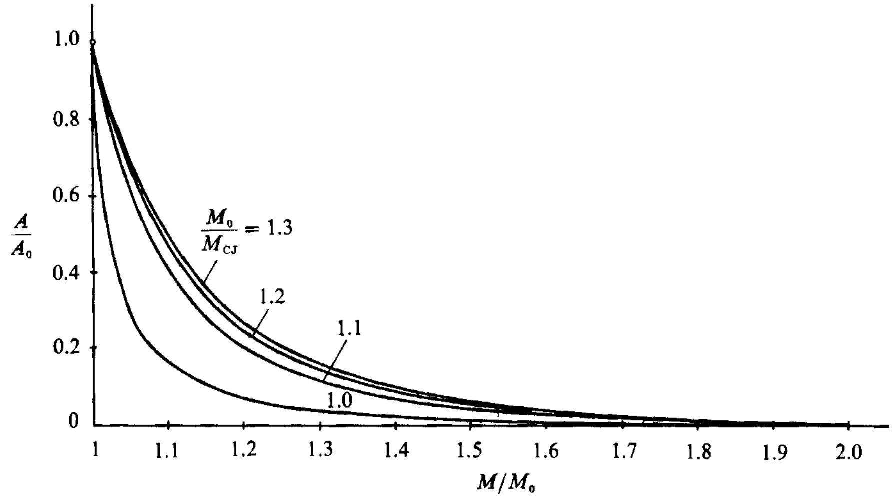

FIGURE 4. Area ratio A/A_0 against the Mach number ratio M/M_0 for different values of M_0. $\gamma_2/\gamma_1 = 0.8$, $Q = 8.1$, $M_{CJ} = 6.055$.

For the special cases of cylindrical and spherical flow, the detonation Mach number can be expressed as a function of the distance x through the result

$$\frac{x}{h_0} = \left[1 - \left(\frac{A(x)}{A_0}\right)^{\frac{1}{\sigma}}\right] \frac{1}{\tan \alpha_w} \tag{3.17}$$

where $\sigma = 1$ for cylindrical flow and $\sigma = 2$ for spherical flow. α_w is the wall angle and $2h_0$ is the channel height at $x = 0$.

4. The propagation of detonation waves in a diverging channel

When a self-sustained CJ-detonation propagates into a channel with increasing cross-section (figure 5), the detonation wave will be decelerated. Since a CJ-detonation is the lower limiting case of a detonation wave, it can no longer exist in the classical form. The diminishing temperature behind the weakening shock front causes a rapid increase of the induction time, and subsequently a separation of the reaction zone.

As long as the temperature of the gas in the induction zone exceeds the self-ignition temperature, the velocity of the reaction zone is prescribed by the shock wave. Therefore, we still have a certain type of coupled system which may be considered as a non-classical detonation. Only if the temperature in the induction zone becomes lower than the self-ignition temperature is the velocity of the reaction zone governed by the laws of flame propagation. Reaction zone and shock wave are now completely decoupled. The detonation wave has ceased to exist.

Generally the reaction zone is fairly thin. Therefore, it is advisable to employ the two-front model as described in §2, which will be a good approximation for our considerations. Furthermore, experiments (Bartlmä & Schröder 1986) have revealed a fact of great importance for the theoretical modelling: in the case of a detached reaction zone, the leading shock wave behaves as it would if the chemical reaction were not present. This gives us the convenient possibility of calculating the shock wave propagation first, thereafter the ignition delay time, and finally the reaction front.

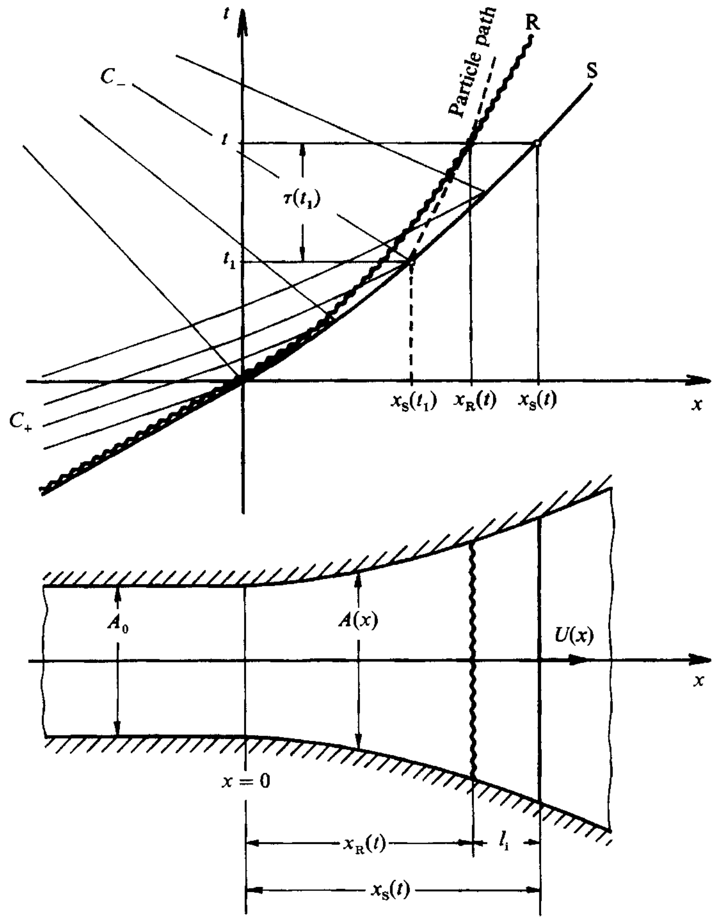

FIGURE 5. Decelerated detonation wave in a diverging channel.

The relation between shock Mach number and the area of the channel cross-section is given by the well-known Chisnell (1975) formula

$$\frac{A(x_s)}{A_0} = \frac{\exp\left\{-\int G(M)\,\mathrm{d}M\right\}}{\exp\left\{-\int G(M_0)\,\mathrm{d}M_0\right\}} = \frac{f(M)}{f(M_0)}, \tag{4.1}$$

where now $M = U/c_0$ and $U(x)$ is the propagation velocity of the shock wave with respect to a fixed coordinate system. The Mach number function $G(M)$ is given by (3.11).

The A–M relation may be obtained immediately by integrating (4.1). The exact solution can be found in Chisnell's paper. Here we give a more convenient approximate formula for $M^2 \gg 1$:

$$\frac{A(x_s)}{A_0} \approx \left(\frac{M}{M_0}\right)^{-\lambda}, \tag{4.2}$$

where
$$\lambda = 1 + \frac{2}{\gamma_1} + \left(\frac{2\gamma_1}{\gamma_1 - 1}\right)^{\frac{1}{2}}, \quad M = \frac{U}{c_0}.$$

Equation (4.2) is formally the same as (3.16); λ has also been assumed to be constant, which is well justified for the shock Mach numbers that usually occur in detonation problems.

From the shock relation we know the state of the unburned gas mixture immediately behind the leading shock at every instant, and also the induction time τ, provided $A(x)$ is given.

In order to continue our considerations we have to specify a particular gas mixture. For a propane–oxygen–argon mixture, τ is given by Burcat *et al.* (1971) as

$$\tau = K[C_3H_8]^{n_1}[O_2]^{n_2}[Ar]^{n_3}\exp\left\{\frac{E}{RT}\right\} \quad s, \tag{4.3}$$

where $n_1 = 0.57$, $n_2 = -1.22$, $n_3 = 0$, and $K = 4.4\times10^{-14}$. Here $[C_3H_8]$, $[O_2]$, $[Ar]$ are the molar concentrations, E is the activation energy and R the universal gas constant.

In addition we restrict ourselves to cylindrical or spherical flow. Using (4.2) we obtain the following relation between the position of the shock front x_s, the channel area, and the shock Mach number:

$$\frac{x_s}{h_0} = \left[\left(\frac{A}{A_0}\right)^{\frac{1}{\sigma}}-1\right]\frac{1}{\tan\alpha_w} = \left[\left(\frac{M}{M_0}\right)^{-\frac{\lambda}{\sigma}}-1\right]\frac{1}{\tan\alpha_w} \tag{4.4}$$

where $\sigma = 1$ for the cylindrical case and $\sigma = 2$ for the spherical case. Again α_w is the wall angle and $2h_0$ the channel height at $x = 0$. Similarly, a relation between the time-coordinate and the shock Mach number can be obtained:

$$\frac{c_0 t}{h_0} = \frac{\lambda}{(\sigma+\lambda)}\frac{1}{M_0\tan\alpha_w}\left[\left(\frac{M}{M_0}\right)^{-\frac{\sigma+\lambda}{\sigma}}-1\right]. \tag{4.5}$$

The thickness of the induction zone at a time t is defined as

$$l_i(t) = x_s(t) - x_R(t). \tag{4.6}$$

For a given time t_1, we know the shock Mach number M_1, the state of the gas mixture behind the shock wave and the corresponding induction time $\tau(t_1)$, given by (4.3). Then, the induction zone thickness $l_i(t)$ at a time $t = t_1+\tau(t_1)$ follows from simple geometrical considerations (figure 5):

$$l_i(t) = x_s(t) - x_s(t_1) - \hat{u}\tau(t_1), \tag{4.7}$$

where $\hat{u}$ is the velocity (assumed to be a constant to good approximation) of a fluid particle within the induction zone. Therefore, $\hat{u}$ is given by the flow velocity behind the shock wave at the time $t = t_1$:

$$\frac{\hat{u}}{c_0} = \frac{\hat{u}_1}{c_0} = \frac{2}{\gamma_1+1}\frac{(M_1^2-1)}{M_1} \approx \frac{2}{\gamma_1+1}M_1. \tag{4.8}$$

Finally we obtain an equation for $l_i(t)$ in dimensionless form:

$$\frac{l_i(t)}{h_0} = \frac{x_s(t)}{h_0} - \frac{x_s(t_1)}{h_0} - \frac{2}{\gamma_1+1}M_1\frac{c_0\tau(t_1)}{h_0}. \tag{4.9}$$

Thus, we may calculate the position of the reaction front step by step from the position of the leading shock wave, as long as the temperature in the induction zone exceeds the self-ignition temperature.

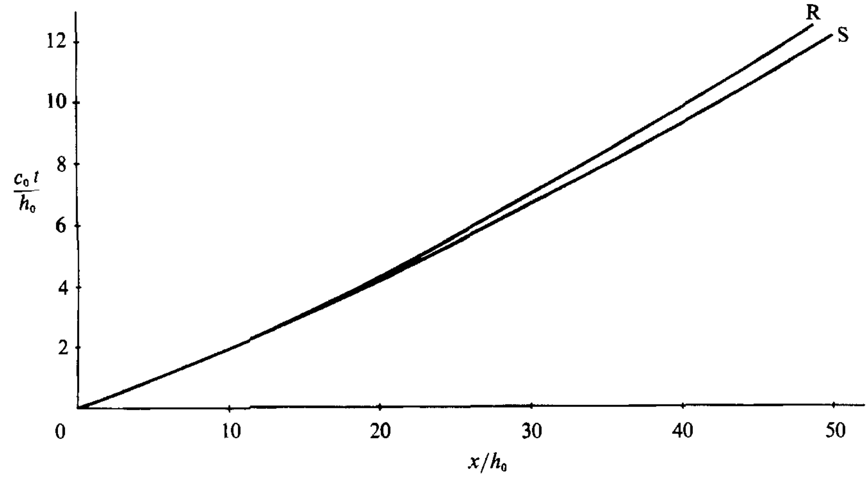

FIGURE 6. Shock wave and reaction front in the (x, t)-plane for a propane–nitrogen–oxygen mixture (cylindrical case).

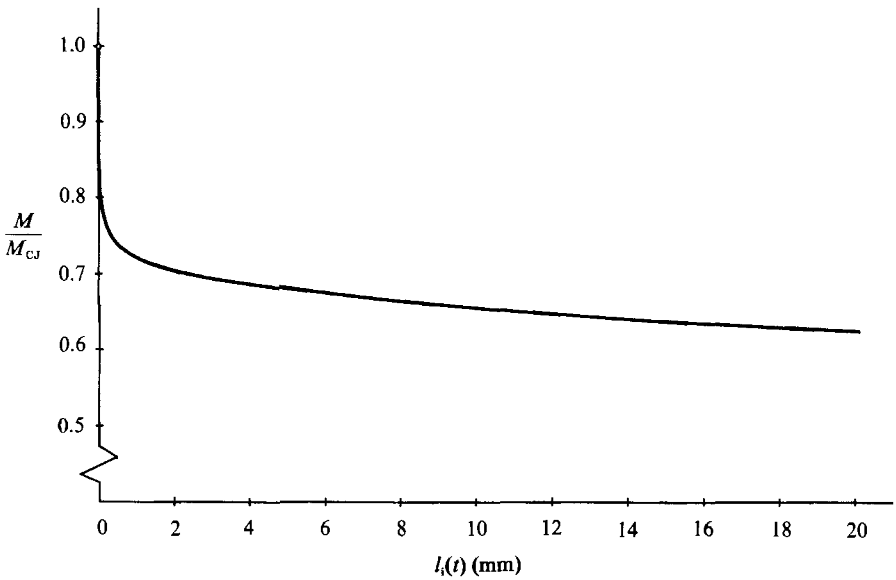

FIGURE 7. Induction zone length l_i in mm as a function of the shock Mach number $M = U/c_0$ for the mixture of figure 6.

The results of such a calculation for a mixture of 8% propane, 40% oxygen, and 52% argon are presented in figures 6 and 7. The incident detonation wave is a CJ-detonation. Cylindrical flow is considered, with a wall angle of $\alpha_w = 15°$. Figure 6 shows the separation of the reaction zone in the (x, t)-plane. Figure 7 demonstrates the exponential increase of the induction zone length with decreasing shock Mach number.

5. Concluding remarks

In considering the problem of detonation wave propagation in channels of varying cross-section, we have to distinguish between two different cases: accelerated detonation waves in converging channels and decelerated detonation waves in diverging channels.

In the first case we always have to deal with overcompressed detonation waves, even if the oncoming wave is a CJ-detonation. Therefore, the situation is relatively simple. A single-front model for the detonation wave and simplified equations (because $M^2 \gg 1$) can be applied. In general it is not possible to describe the propagation process by the leading shock of the detonation wave alone. Only for highly overcompressed detonation waves does the influence of the chemical reaction vanish. Therefore, a suitable extension of Chisnell's theory is necessary.

The second case, decelerated detonation waves, is entirely different, especially if the incident wave is a CJ-detonation. With decreasing wave Mach number the induction time for the chemical reaction becomes more and more important. Subsequently, separation of the leading shock and the reaction zone occurs. For the analytical treatment a two-front model will now be appropriate. As long as the temperature behind the shock exceeds the self-ignition temperature of the mixture, the velocity of the reaction front is still governed by the shock wave, and we may consider the system as a non-classical detonation. In certain cases, however, the calculation can be considerably simplified by the empirical result that the detached shock wave behaves in a way that is independent of the chemical reaction. Good agreement between the experiments and the results of Whitham's approach, containing the one-dimensional theory, can be achieved.

However, it should be remembered that Chisnell's theory neglects all re-reflected waves, and no really satisfactory explanation has been found up to now for its amazing accuracy in many cases, despite its simplicity. We also should keep in mind that there might be cases of decelerated detonation waves where the interactions between the reaction wave and the shock accumulate to an intolerable extent.

Not so clear at present is the case of a decelerated overcompressed detonation wave. The question arises as to whether the latter can be decelerated until the state of a CJ-detonation is attained, or whether separation of shock and reaction zone occurs earlier for instability reasons. The answer has yet to be given by experiments. In any case, an analytical treatment is possible, without difficulty, using the theoretical models given above.

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